Resolution of the Ehrenfest Paradox in the Dynamic Interpretation of Lorentz Invariance

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Z. Naturforsch. **53a**, 751–754 (1998); received July 14, 1998

In the dynamic Lorentz-Poincaré interpretation of Lorentz invariance, clocks in absolute motion through a preferred reference system (resp. aether) suffer a true contraction and clocks, as a result of this contraction, go slower by the same amount. With the one-way velocity of light unobservable, there is no way this older pre-Einstein interpretation of special relativity can be tested, except in cases involving rotational motion, where in the Lorentz-Poincaré interpretation the interaction symmetry with the aether is broken.

In this communication it is shown that Ehrenfest's paradox, the Lorentz contraction of a rotating disk, has a simple resolution in the dynamic Lorentz-Poincaré interpretation of Lorentz invariance and can perhaps be tested against the prediction of special relativity.

One of the oldest and least understood paradoxes of special relativity is the Ehrenfest paradox [1], the Lorentz contraction of a rotating disk, whereby the ratio of circumference of the disk to its diameter should become less than π. The paradox gave Einstein the idea that in accelerated frames of reference the metric is non-euclidean and that by reason of his principle of equivalence the same should be true in the presence of gravitational fields. The disk to become non-euclidean, though, can only mean that it becomes curved, for example deformed into a dish, as it was conjectured by Ives [2], but this would not work if the disk is replaced by a rod. The different attempts to resolve the Ehrenfest paradox, none of them satisfactory, have been summarized by Phipps [3].

There are two other experimentally verified effects which seem to contradict the kinematic interpretation of Lorentz invariance: The Sagnac effect and unipolar induction. In all of these cases rotational motion is involved. Whereas for rectilinear acceleration the requirement of Born-rigidity in special relativity [4] can be sustained by a proper acceleration program for different parts of a finite size body, this is not possible for rotational motion.

It is the purpose of this communication to show that the Ehrenfest paradox can be easily explained with the dynamic pre-Einstein theory of relativity by Lorentz and Poincaré. It assumes the existence of a preferred reference system (or aether), with all rods in absolute motion against the preferred reference system to suffer a true contraction. With all clocks understood as light clocks made up of Lorentz-contracted rods, clocks in absolute motion go slower by the same amount. For an observer in absolute motion against the preferred reference system, the contraction and time dilation observed on an object at rest in this reference system is there explained as an illusion caused by the Lorentz contraction and time dilation of the moving observer, not the observed object at rest. The reason for this symmetry, giving always the same impression for an observer in relative motion against an object, is the group structure of the Lorentz transformations. It makes it impossible to measure the one-way velocity of light, and all claims to the contrary are false [5].

In Einstein's theory, Lorentz invariance is a purely kinematic space-time symmetry, explaining the Lorentz contraction as a rotation in space-time. In the Lorentz-Poincaré theory the contraction is real and caused by the interaction with an aether. Both interpretations give identical results except in the case of rotational motion. This underlines the importance of experiments involving rotational motion, because they may be capable to decide between the Einstein and Lorentz-Poincaré theory [6].

In the Lorentz-Poincaré theory the atoms making up the arms of the Michelson-Morely interferometer, for example, are held together by electromagnetic forces. These forces are derived from potentials which in the aether rest frame obey the inhomogeneous wave equation

$$-\frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} + \nabla^2 \Phi = -4\pi\rho(r,t). \qquad (1)$$

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For a body in static equilibrium (1) becomes

$$\nabla^2 \Phi = -4\pi\rho(r) \tag{2}$$

with the source $\rho(r)$ in the body (resp. the arms of the interferometer).

To obtain the corresponding equation after the body has been set into absolute motion with the velocity v into the x-direction, (1) must be Galilei-transformed by

$$x' = x - vt,\ y' = y,\ z' = z,\ t' = t, \tag{3}$$

whereby it becomes

$$-\frac{1}{c^2}\frac{\partial^2 \Phi'}{\partial t'^2} + \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 \Phi}{\partial x'^2} + \frac{\partial^2 \Phi'}{\partial y'^2} + \frac{\partial^2 \Phi'}{\partial z'^2} = -4\pi\rho'(r',t'). \tag{4}$$

After the body has settled into a new state of static equilibrium, one has $\partial/\partial t' = 0$, $y' = y$, $z' = z$, and one finds

$$\left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2 \Phi}{\partial x'^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2} = -4\pi\rho'(x', y, z). \tag{5}$$

Comparing (5) with (2) demonstrates that (2) can be obtained from (5) setting $\mathrm{d}x' = \mathrm{d}x\sqrt{1 - v^2/c^2}$, implying a uniform contraction of the body and slower going clocks. The forces transmitted by the electromagnetic potentials causing an attraction between the atoms of the body must be balanced by a repulsive force to prevent the body from collapsing. In quantum mechanics these repulsive forces come from the zero point energy fluctuations. In the Lorentz-Poincaré theory they come from fluctuations of an electromagnetic background field. According to Einstein and Hopf [7], a point particle of mass m and electric charge e, moving with the velocity v through an isotropic electromagnetic radiation field with the frequency spectrum $f(\omega)$, suffers the friction force

$$F = -\frac{4}{5}\frac{\pi^2 e^2}{mc^2}\left[f(\omega) - \frac{\omega}{3}\frac{\mathrm{d}f(\omega)}{\mathrm{d}\omega}\right]v. \tag{6}$$

According to a variational principle by Helmholtz, the frictional dissipation of a fluid flow should assume a minimum, and it is plausible that this should be also true for the radiation field. It therefore should have the form

$$f(\omega) = \text{const. } \omega^3, \tag{7}$$

for which the friction force vanishes. Now, not only is (7) Lorentz invariant, but at the same time it also has the form of the quantum mechanical zero point energy spectrum. For this reason it can provide the Lorentz invariant repulsive force to prevent matter from collapsing.

Returning to the Ehrenfest paradox we first consider the elastic deformation of a solid rotating cylinder, ignoring Lorentz contraction. The solution of this problem can be found in the textbook literature [8] and is given by

$$\varepsilon = \frac{\rho\omega^2(1+\mu)(1-2\mu)}{8E(1-\mu)}\, r\left[(3-2\mu)R^2 - r^2\right], \tag{8}$$

where $\varepsilon = \varepsilon(r)$ is the radial deformation of the rotating cylinder of radius R and density ρ, ω is its angular velocity, E the modul of elasticity and μ the Poisson number. Equation (8) is obtained from the static equation for an elastic body under the influence of the force density ρa. Expressed in general curvilinear coordinates, this equation is

$$\sigma^k_{i;k} = \rho a_i \tag{9}$$

where σ^i_k is the stress tensor in mixed covariant-contravariant form and where the colon stands for the covariant derivative. In polar coordinates r, ϕ, with $\rho a = \rho\omega^2 r$, (9) is

$$r\frac{\mathrm{d}\sigma^r_r}{\mathrm{d}r} + \sigma^r_r - \sigma^\phi_\phi = -\rho\omega^2 r^2. \tag{10}$$

Expressing the stress tensor σ^i_k by the stress strain relations one obtains the solution (8).

Next we include the effect of Lorentz contraction. Because of Lorentz invariance under linear uniform motion we may consider the rotating cylinder to be at rest in the preferred aether rest frame. The Lorentz contraction acting in the ϕ-direction leads to a relative change in length of a line element attached to the rotating cylinder, given by

$$\frac{\delta s}{s} = \frac{s - s_0}{s_0} = \sqrt{1 - v^2/c^2} - 1 \approx -\frac{1}{2}\frac{v^2}{c^2} = -\frac{\omega^2}{2c^2}r^2. \tag{11}$$

In the dynamic interpretation of Lorentz invariance it is caused by a reduction in the attractive force between adjacent atoms, resulting in a strain in the opposite direction. In computing the deformation of the rotating rod we first assume that the Poisson number is equal to zero, presenting the limit of maximum rigidity. In this limit there is no lateral stress, with the additional azimuthal stress given by

$$\sigma^\phi_\phi = E\varepsilon^\phi_\phi = -E\frac{\delta s}{s} = \frac{E\omega^2}{2c^2}r^2. \tag{12}$$

Adding this stress to σ_ϕ^ϕ on the l.h.s. of (10) one finds

$$r\frac{d\sigma_r^r}{dr}+\sigma_r^r-\sigma_\phi^\phi=-\rho\omega^2r^2\left(1-\frac{E}{2\rho c^2}\right). \quad (13)$$

One therefore has to replace in (8) ω^2 by $\omega^2(1-E/2\rho c^2)$, which for $\mu=0$ leads to

$$\varepsilon=\frac{\rho\omega^2(1-E/2\rho c^2)}{8E}\left[r(3R^2-r^2)\right], \quad (14)$$

while for the centrifugal force alone it would give

$$\varepsilon=(\rho\omega^2/8E)[r(3R^2-r^2)]. \quad (15)$$

To isolate the effect of the Lorentz contraction, one has to substract (15) from (14) with the result that

$$\varepsilon_L(r)=-(\omega^2/16c^2)\,r(3R^2-r^2). \quad (16)$$

For $r=R$ this yields

$$\varepsilon_L(R)=-\omega^2R^3/8c^2. \quad (17)$$

Accordingly, the radius of the cylinder is reduced from its value $R=R_0$ prior to being set into rotational motion to

$$R=R_0-\omega^2R^3/8c^2$$
$$\approx R_0(1-\omega^2R_0^2/8c^2) \quad (18)$$

with the circumference of the rim reduced by 1/4 the amount $(1-\omega^2R_0^2/c^2)^{1/2}-1$ Lorentz contraction predicts. This result, valid only in the limit $\mu=0$, was previously obtained by Lorentz [9] and Eddington [10] under some ad hoc assumptions.

For a Poisson number $\mu\neq0$, the azimuthal strain caused by the Lorentz contraction in the ϕ-direction leads to an additional stress in the r-direction, but with a different stress in the ϕ-direction. Applying the strain stress relations [8] to this case, one obtains

$$\sigma_\phi^\phi=\frac{1-\mu}{(1+\mu)(1-2\mu)}E\varepsilon_\phi^\phi$$
$$=\frac{1-\mu}{(1-\mu)(1-2\mu)}\frac{E\omega^2}{2c^2}r^2,$$

$$\sigma_r^r=\frac{\mu}{(1+\mu)(1-2\mu)}E\varepsilon_\phi^\phi$$
$$=\frac{\mu}{(1+\mu)(1-2\mu)}\frac{E\omega^2}{2c^2}r^2. \quad (19)$$

Hence, adding to the l.h.s. of (10) these additional stresses one obtains

$$r\frac{d\sigma_r^r}{dr}+\sigma_r^r-\sigma_\phi^\phi=-\rho\omega^2r^2\left(1-\frac{E}{2\rho c^2(1+\mu)}\right). \quad (20)$$

One therefore has for the total deformation

$$\varepsilon(r)=$$
$$(\rho\omega^2/8E)(1-E/(2\rho c^2(1+\mu))[r(3R^2-r^2)] \quad (21)$$

with the part due only to Lorentz contraction given by

$$\varepsilon_L(r)=-(\omega^2/(16(1+\mu)c^2))[r(3R^2-r^2)], \quad (22)$$

hence

$$\varepsilon_L(R)=-(\omega^2/(8(1+\mu)c^2))R^3. \quad (23)$$

Accordingly, the radius is now decreased from R_0 to R by

$$R=R_0-(\omega^2/(8(1+\mu))c^2)R^3$$
$$\approx R_0(1-\omega^2R_0^2/(8(1+\mu)c^2)), \quad (24)$$

differing from the result of Lorentz and Eddington by the factor $(1+\mu)$.

Finally, we turn to the case of a rotating disk. As long as we are only interested in the effect of the Lorentz stresses, the σ_ϕ^ϕ and σ_r^r components of the stress tensor remain the same, but now one has also to consider the component σ_z^z:

$$\sigma_z^z=\frac{\mu}{(1+\mu)(1-2\mu)}\frac{E\omega^2r^2}{2c^2}. \quad (25)$$

The z-component of the static equation is

$$\frac{\partial\sigma_z^z}{\partial z}=0, \quad (26)$$

hence $\sigma_z^z=f(r)$, with $f(r)$ given by (25). From the strain-stress relation one obtains

$$\varepsilon_z^z=\frac{1}{E}\left[(1+\mu)\sigma_z^z-\mu\left(\sigma_r^r+\sigma_\phi^\phi+\sigma_z^z\right)\right]$$
$$=\frac{1}{E}\left[\sigma_z^z-\mu\left(\sigma_r^r+\sigma_\phi^\phi\right)\right]=0, \quad (27)$$

hence $\partial\varepsilon_z/\partial z=\varepsilon_z^z=0$. Accordingly, there is no deformation of the disk in the z-direction, with the deformation in the r-direction remaining unchanged.

To treat the problem of the rotating disk in the theory of relativity one must make the general relativistic transformations to the noninertial reference system of the rotating disk. This problem had been treated by Berenda [11] with the result that the disk deforms into a surface of negative Gaussian curvature K given by

$$K=-\frac{3\omega^2}{c^2}\left(1-\frac{\omega^2r^2}{c^2}\right)^{-2}. \quad (28)$$

For $\omega^2 r^2/c^2 \ll 1$, one has $K \approx -3\omega^2/c^2$ and the equation of the surface $z = z(r)$ is a pseudosphere satisfying the differential equation

$$\frac{\mathrm{d}r}{\mathrm{d}z} = \frac{\sqrt{3}\omega z/c}{\sqrt{1-3\omega^2 z^2/c^2}} \tag{29}$$

with the solution

$$\sqrt{3}\omega r/c = \sqrt{1-3\omega^2 z^2/c^2} - 1/2 \log\left(\frac{1+\sqrt{1-3\omega^2 z^2/c^2}}{1-\sqrt{1-3\omega^2 z^2/c^2}}\right). \tag{30}$$

It has the form of a cusp with a sharp peak at $r = 0$ where $z = z_0 = (1/3)\,c/\omega$ and $\mathrm{d}z/\mathrm{d}r = -\infty$. For $\omega r/c \ll 1$, (30) has the approximate solution

$$z \approx z_0\sqrt{1-3(\omega r/c)^{2/3}} \approx z_0\left(1-(3/2)\,(\omega r/c)^{2/3}\right). \tag{31}$$

Therefore, the height of the peak at $r = 0$ over the disk surface is approximately given by

$$h \approx (3/2)(\omega r/c)^{2/3} z_0. \tag{32}$$

With centrifuges $\omega r \sim 10^5$ cm/s can be reached, whereby $h/z_0 \approx 3 \times 10^{-4}$, a measurable amount, provided the deformation by the centrifugal force can be subtracted. By comparison, there is no likewise deformation in the Lorentz-Poincaré theory, with the general relativistic treatment giving the full Lorentz contraction $\sqrt{1-v^2/c^2}-1$ for the rim of the disk.

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